Apr 14: Field extensions: simple & algebraic

The quartic question on HW2 was challenging.

§ 0. Recap

Let K be a field

Let $f(x) \in K[x]$ irred poly

FACT $K[x]/(f)$ is a field and $K \to K[x]/(f)$ field extension and degree

$$\underbrace{|K[x]/(f) : K|}_{\text{degree is defined as } \dim_K K[x]/(f)} = \deg f$$

If $K \subset L$ field extension and $\alpha \in L$, then $K(\alpha)$ as the smallest subfield containing both K & α.

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Consider

$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$

$\mathbb{Q} \subset \mathbb{Q}(\sqrt{3})$

Also have $\boxed{\mathbb{Q}(\sqrt{2},\sqrt{3})} \subset \mathbb{C}$

$\sqrt{6} \in$

means smallest subfield containing $\sqrt{2}, \sqrt{3}$

Ques: Is $\mathbb{Q}(\sqrt{2},\sqrt{3})$ simple ext. of $\mathbb{Q}$?

Suggestion: $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$

Certainly $\alpha = \sqrt{2}+\sqrt{3} \in \mathbb{Q}(\sqrt{2},\sqrt{3})$

Observation: $|\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}| = 4$

$\alpha = \sqrt{2}+\sqrt{3}$

$\alpha^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$

$\rightsquigarrow$ $\{1, \alpha, \alpha^2\}$ lin. indep.

because $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ are lin. indep

$$\alpha^3 = (\sqrt{2}+\sqrt{3})(5+2\sqrt{6})$$
$$= 5\sqrt{2} + 2\sqrt{12} + 5\sqrt{3} + 2\sqrt{18}$$
$$= 5\sqrt{2} + 4\sqrt{3} + 5\sqrt{3} + 6\sqrt{2}$$
$$= 11\sqrt{2} + 9\sqrt{3}$$

$\rightsquigarrow$ $\{1, \alpha, \alpha^2, \alpha^3\}$ are lin. indep

Is $\sqrt{2} \in \operatorname{Span}\{1, \alpha, \alpha^2, \alpha^3\}$

$$\frac{\alpha^3 - 9\alpha}{2} = \frac{(11\sqrt{2} + \cancel{9\sqrt{3}}) - 9(\sqrt{2} + \cancel{\sqrt{3}})}{2}$$
$$= \sqrt{2} \quad \checkmark$$

Similarly $\dfrac{\alpha^3 - 11\alpha}{-2} = \sqrt{3}$

Conclusion: Yes, it is simple.
$\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$

Def Let $K \subset L$ be a field ext. We say $\alpha \in L$ is algebraic over K if $\exists f(x) \in K[x]$ such that $f(\alpha) = 0$.

Ex: $\mathbb{Q} \subset \mathbb{C}$

- $\sqrt{2}, \sqrt{3} \in \mathbb{C}$ are algebraic over $\mathbb{Q}$
- $i \in \mathbb{C}$ algebraic$/\mathbb{Q}$
- $e, \pi \in \mathbb{C}$ are not algebraic$/\mathbb{Q}$ (takes work to show this)
- $e, \pi \in \mathbb{C}$ are algebraic$/\mathbb{R}$ $(x - e$ or $x - \pi)$
- $i, \pi \in \mathbb{C}$ are algebraic$/\mathbb{R}$ (but not $\mathbb{Q}$)

Proposition Let $K \subset L$ field ext. Let $\alpha \in L$ be algebraic over K. Then there exists a unique monic irreducible polynomial $f(x) \in K[x]$ with $f(\alpha) = 0$.

Observation: If α is a root of $f(x)$, then α is also root of $f(x)g(x)$ for any $g \in K[x]$.

Proof

$I = \{ f(x) \in K[x] \mid f(\alpha) = 0 \} \subset K[x]$

Claim: I is an ideal

True b/c observation. $(\forall f \in I, g \in K[x]$, $fg \in I)$

Know $K[x]$ is a principal ideal domain, i.e. every ideal $J \subset K[x]$ is of the form $J = (h)$ for $h \in K[x]$

- **Proposition** Let $K \subset L$ field ext. Let $\alpha \in L$ be algebraic over K. Then there exists a unique monic irreducible polynomial $f(x) \in K[x]$ with $f(\alpha) = 0$.

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Know $K[x]$ is a principal ideal domain, i.e. every ideal $J \subset K[x]$ is of the form $J = (h)$ for $h \in K[x]$

Therefore $\exists$ some $f(x) \in K[x]$ such that

$$I = (f)$$

- $f \in I \Rightarrow \boxed{f(\alpha) = 0}$
- By dividing by leading coeff, can assume f is monic.
- Why is f irreducible?

If not, $f = f_1 \cdot f_2$

$0 = f(\alpha) = f_1(\alpha) f_2(\alpha)$

$\Rightarrow \alpha$ is a root of either f_1, f_2

Assume $f_1(\alpha) = 0$

$I = (f) \subset (f_1) \subset I$

$\Rightarrow (f) = (f_1) \Rightarrow f_2$ is a scalar.

- Why is f unique?

For any other $g(x)$ with $g(\alpha) = 0$, then $f \mid g$.

B/c f monic, has to be unique.

: **Proposition** Let $K \subset L$ field ext. Let $\alpha \in L$ be algebraic over K. Then there exists a unique monic irreducible polynomial $f(x) \in K[x]$ with $f(\alpha) = 0$.

We call $f(x)$ the minimal polynomial of α over K

Ex:

- $\mathbb{Q} \subset \mathbb{C}$, $\sqrt{2}$
  min poly $\sqrt{2}$ over $\mathbb{Q}$ = $x^2 - 2$
- $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$
  min poly $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$ = $x^2 - 3$
- What $\alpha = \sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$
- Min poly of α over $\mathbb{Q}$

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$\alpha = \sqrt{2} + \sqrt{3}$

$\alpha^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$

$\alpha^3 = 11\sqrt{2} + 9\sqrt{3}$

Know $\{1, \alpha, \alpha^2, \alpha^3\}$ lin. indep.

$\Longrightarrow \nexists$ cubic $f(x)$ with $f(\alpha) = 0$

$\alpha^4 = (\alpha^2)^2 = (5 + 2\sqrt{6})^2 = 25 + 4 \cdot 6 + 20\sqrt{6}$

$= 49 + 20\sqrt{6}$

$\alpha^4 - 10\alpha^2 = 49 - 50 = -1$

$\Longrightarrow \boxed{\alpha^4 - 10\alpha^2 + 1 = 0}$

$\Longrightarrow f(x) = x^4 - 10x^2 + 1$ is the min poly of α over $\mathbb{Q}$

Ques: What is min poly of $\alpha = \sqrt{2}+\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$?

$g(x) = x^3 - 9x - 2\sqrt{2}$ $\quad g(\alpha) = 0$

$\alpha^2, \alpha, 1$ should be lin dep. over $\mathbb{Q}(\sqrt{2})$

$\sqrt{2}\cdot\alpha = 2+\sqrt{6}$

$$\boxed{\alpha^2 - 2\sqrt{2}\alpha - 1 = 0}$$

$\Rightarrow f(x) = x^2 - 2\sqrt{2}x - 1$

min poly

$|\mathbb{Q}(\alpha):\mathbb{Q}| = \deg$ of min poly of α over $\mathbb{Q}$

$\alpha = \sqrt{2}+\sqrt{3}$

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$\Rightarrow f(x) = x^4 - 10x^2 + 1$ is the min poly of α over $\mathbb{Q}$